Chiral zero modes of the *SU*(*n*) Wess–Zumino–Novikov–Witten model

2003 J. Phys. A: Math. Gen. 36 3855

(http://iopscience.iop.org/0305-4470/36/13/316)

Institute of Physics Publishing | Journal of Physics A: Mathematical and General
J. Phys. A: Math. Gen. **36** (2003) 3855–3875 PII: S0305-4470(03)57010-X

Chiral zero modes of the $SU(n)$ Wess–Zumino–Novikov–Witten model

P Furlan[1,2], **L K Hadjiivanov**[1,2,3] **and I T Todorov**[3,4]

[1] Dipartimento di Fisica Teorica dell' Università degli Studi di Trieste, Strada Costiera 11, I-34014 Trieste, Italy
[2] Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Trieste, Trieste, Italy
[3] Theoretical Physics Division, Institute for Nuclear Research and Nuclear Energy, Tsarigradsko Chaussee 72, BG-1784 Sofia, Bulgaria
[4] Erwin Schrödinger International Institute for Mathematical Physics, Boltzmanngasse 9, A-1090 Wien, Austria

E-mail: furlan@trieste.infn.it, lhadji@inrne.bas.bg and todorov@inrne.bas.bg

Received 2 December 2002, in final form 6 February 2003
Published 19 March 2003
Online at stacks.iop.org/JPhysA/36/3855

Abstract

We define the chiral zero modes' phase space of the $G = SU(n)$ Wess–Zumino–Novikov–Witten (WZNW) model as an $(n-1)(n+2)$-dimensional manifold $\mathcal{M}_q$ equipped with a symplectic form Ω_q involving a Wess–Zumino term ρ which depends on the monodromy M and is implicitly defined (on an open dense neighbourhood of the group unit) by

$$\mathrm{d}\rho(M) = \frac{1}{3}\,\mathrm{tr}(M^{-1}\,\mathrm{d}M)^3. \qquad (*)$$

This classical system exhibits a Poisson–Lie symmetry that evolves upon quantization into a $U_q(s\ell_n)$ symmetry for q a primitive even root of 1. For each (non-degenerate, constant) solution of the classical Yang–Baxter equation we write down explicitly a $\rho(M)$ satisfying equation $(*)$ and invert the form Ω_q, thus computing the Poisson bivector of the system. The resulting Poisson brackets (PB) appear as the classical counterpart of the exchange relations of the quantum matrix algebra studied previously in Furlan *et al* (2000 *Preprint* hep-th/0003210). We argue that it is advantageous to equate the determinant D of the zero modes' matrix $\left(a^j_\alpha\right)$ to a pseudoinvariant under permutations q-polynomial in the $SU(n)$ weights, rather than to adopt the familiar convention $D = 1$. A finite-dimensional 'Fock space' operator realization of the factor algebra $\mathcal{M}_q/\mathcal{I}_h$, where $\mathcal{I}_h$ is an appropriate ideal in $\mathcal{M}_q$ for $q^h = -1$, is briefly discussed.

PACS numbers: 02.20.Uw, 02.10.Hh, 02.40.Vh
Mathematics Subject Classification: 17B37, 53D05, 81S10

1. Introduction

Two-dimensional conformal current algebra models are known to lead to an unconventional problem of classical symplectic dynamics defined in terms of a multivalued action [49, 44, 50] or, equivalently, by a closed—but not exact—3-form [35], depending on a group-valued field. It has been noted at an early stage of this development [1] that the most interesting new features of the theory already appear in a finite-dimensional 'toy model'. The present paper is devoted to a study of a version of such a finite-dimensional 'chiral zero modes' model. We display its precise relation to the (infinite-dimensional) WZNW theory, reformulate it as a constrained dynamical system in the case when the underlying group is $SU(n)$, compute Poisson brackets among the basic dynamical variables for a given non-degenerate solution of the classical (dynamical) Yang–Baxter equations and demonstrate that they appear as a (quasi)classical limit of quantum exchange relations considered earlier [32].

1.1. The zero modes' manifold

Let G be a semisimple compact Lie group of $n \times n$ matrices with Lie algebra $\mathcal{G}$. The zero modes' manifold of a chiral WZNW model is not uniquely determined by the corresponding two-dimensional (2D) conformal theory. It depends on the splitting of the G-valued field $g(x^0, x^1)$ into chiral factors,

$$g(x^0, x^1) = g_L(x^1 + x^0) g_R^{-1}(x^1 - x^0) \tag{1.1}$$

which obey a twisted periodicity condition (involving monodromy degrees of freedom),

$$g_C(x + 2\pi) = g_C(x) M \qquad C = L, R \qquad M \in G \tag{1.2}$$

implying that the 2D field is periodic: $g(x^0, x^1 + 2\pi) = g(x^0, x^1)$. A further arbitrariness is involved in the factorization of the chiral fields $g_C(x)$ into (classical counterparts of) *chiral vertex operators* $u(x)$ and *zero modes* a; we shall write, in particular, the left movers' field in the form

$$g_L(x)^A_\alpha = u(x)^A_j a^j_\alpha \qquad (A, j, \alpha = 1, \ldots, n). \tag{1.3}$$

The chiral vertex operators have, by definition, diagonal monodromies so that the (x-independent) matrix $a = \left(a^j_\alpha\right)$ is chosen to diagonalize M:

$$aM = M_p a \qquad M_p = \overline{q}^{2\hat{p}} \qquad q = \mathrm{e}^{-\mathrm{i}\frac{\pi}{k}} \qquad \overline{q} = \mathrm{e}^{\mathrm{i}\frac{\pi}{k}}. \tag{1.4}$$

Here k is the *Kac–Moody level* appearing as a coupling constant in the WZNW model [50] and $\hat{p}$ is a diagonal matrix whose entries define a weight vector belonging to the Weyl alcove $\mathcal{A}_n$ of the dual to the Cartan subalgebra of $\mathcal{G}$. For $\mathcal{G} = su(n)$

$$\hat{p} = \begin{pmatrix} p_1 & 0 & \ldots & 0 \\ 0 & p_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & p_n \end{pmatrix} \tag{1.5}$$

and the Weyl alcove can be conveniently identified with

$$\mathcal{A}_n = \left\{ p = \{p_i\}_{i=1}^n,\ p_{ij} := p_i - p_j > 0 \text{ for } i < j,\ P := \frac{1}{n}\sum_{i=1}^n p_i = 0 \right\} \tag{1.6}$$

p_j playing thus the role of barycentric coordinates.

While the weights p_{α_j} corresponding to the simple roots α_j of $\mathcal{G}$ ($p_{\alpha_j} = p_{j j+1}$ for $\mathcal{G} = su(n)$) provide an intrinsic characteristic of the state space of (both the chiral and the 2D)

WZNW model, the zero mode matrix a^j_α is gauge dependent. We shall use this freedom to work in a 'covariant but not unitary gauge' (discussed in section 3) and to equate, for $G = SU(n)$, the determinant D of $\left(a^j_\alpha\right)$ to a pseudo-invariant under permutations of the p_j function of p (cf [32]),

$$D := \det\left(a^j_\alpha\right) = \mathcal{D}_q(p) := \prod_{i<j}[p_{ij}] \qquad \text{for} \quad G = SU(n)$$
$$[p] := \frac{q^p - \overline{q}^p}{q - \overline{q}} \qquad (q\overline{q} = 1) \tag{1.7}$$

rather than to 1 as done in most related studies [1, 4, 8, 15, 9, 10, 19].

Remark 1.1. We use on purpose different notation for the indices such as A, j, α of u and a that run in the same range (1.3) since they have rather different nature. While the chiral model is invariant under left shifts of G (acting on A), it only admits a Poisson–Lie (or quantum group) symmetry with respect to α, while j labels the diagonal elements of M_p.

1.2. The case $n = 2$ and its $k \to \infty$ limit: the form Ω_q for $SU(2)$

The advantage of the ansatz (1.7) (as compared to the conventional $D = 1$) is exhibited on the simple example of the $SU(2)$ model space and its q-deformation which we proceed to sketch. It can also be viewed as an introduction to the general case.

The realization of all irreducible representations (IR) of $SU(2)$ with multiplicity 1 in the Fock space of a pair of creation and annihilation operators is half a century old (see [46, 12]). Its classical counterpart is the space $\mathbb{C}^2$ regarded as a Kähler manifold with a symplectic form

$$\Omega_1 = (\mathrm{i}\mathrm{d}z_\alpha \wedge \mathrm{d}\bar{z}^\alpha \equiv)\, \mathrm{i}\mathrm{d}z_\alpha\, \mathrm{d}\bar{z}^\alpha \equiv \mathrm{i}(\mathrm{d}z_1\, \mathrm{d}\bar{z}^1 + \mathrm{d}z_2\, \mathrm{d}\bar{z}^2). \tag{1.8}$$

(We omit throughout this paper the wedge sign $\wedge$ for the exterior product of differentials but keep it for the skew product of vector fields.) The corresponding Poisson bivector,

$$\mathcal{P}_1 = \mathrm{i}\frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial \bar{z}^\alpha} \tag{1.9}$$

yields the PB counterpart of the canonical commutation relations for (bosonic) creation and annihilation operators:

$$\{z_1, z_2\} = 0 = \{\bar{z}^1, \bar{z}^2\} \qquad \{z_\alpha, \bar{z}^\beta\} = \mathrm{i}\delta^\beta_\alpha. \tag{1.10}$$

In order to express Ω_1 (1.8) in terms of the above 'group like' variable $a = \left(a^j_\alpha\right)$ and 'weight' $p \equiv p_{12}$, we set

$$a = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}^2 & \bar{z}^1 \end{pmatrix} \qquad p := \det a = z_1\bar{z}^1 + z_2\bar{z}^2 \quad (>0 \Leftrightarrow p \in \mathcal{A}_2) \tag{1.11}$$

$$\hat{p} = \tfrac{1}{2}\sigma_3 p. \tag{1.12}$$

A simple calculation allows us then to rewrite Ω_1 as an exact 2-form:

$$\Omega_1 = -\mathrm{i}\,\mathrm{d}\,\mathrm{tr}(\hat{p}\, \mathrm{d}a a^{-1}). \tag{1.13}$$

The symplectic form Ω_q for the $SU(2)_k$ WZNW zero modes (derived for the general $SU(n)_k$ case in section 2) appears as a one-parameter deformation of (1.13):

$$\Omega_q(a, M_p) = \frac{k}{4\pi}\left\{\mathrm{tr}\big(\mathrm{d}a a^{-1}\left(2\,\mathrm{d}M_p M_p^{-1} + M_p\, \mathrm{d}a a^{-1} M_p^{-1}\right)\big) - \rho(a^{-1}M_p a)\right\}. \tag{1.14}$$

Here M_p is the diagonal matrix defined in (1.4), and ρ is the WZ term:

$$\frac{k}{2\pi}\,\mathrm{d}M_p M_p^{-1} = \mathrm{i}\,\mathrm{d}\hat{p} \qquad \mathrm{d}\rho(M) = \frac{1}{3}\,\mathrm{tr}(\mathrm{d}MM^{-1})^3. \tag{1.15}$$

(The 3-form on the right-hand side is closed but not exact on G; the complex 2-form ρ can only be defined on an open dense neighbourhood G_0 of the identity of G.)

The phase space $\mathcal{M}_q$ is a four-dimensional surface in the five-dimensional space of variables a^j_α and p, singled out by equation (1.7):

$$(\det a \equiv) \quad D = [p] \quad (\to p \text{ for } k \to \infty, \text{ resp.}\, q \to 1). \tag{1.16}$$

To summarize, for (undeformed) $SU(2)$ creation and annihilation operators the determinant (1.11) plays the role of a number operator. More precisely, in the quantum theory $p \in \mathbb{N}$ is the dimension of the IR of $SU(2)$ spanned by all homogeneous polynomials of the creation operators a^1_α of degree $p-1$ (acting on the Fock space vacuum). For $p > 0$ we can introduce new matrix variables with determinant 1,

$$g^j_\alpha := \frac{1}{\sqrt{p}} a^j_\alpha \qquad \det\left(g^j_\alpha\right) = 1 \tag{1.17}$$

preserving the form of Ω_1 $(=-\mathrm{i}\,\mathrm{d}\,\mathrm{tr}(\hat{p}\,\mathrm{d}gg^{-1}))$. The new variables $\left(g^j_\alpha\right)$ obeying (1.17), however, would not satisfy the canonical PB relations for creation and annihilation operators. For $q \neq 1$ (k finite) a change of variables $a^j_\alpha \to g^j_\alpha = [p]^{-1/2} a^j_\alpha$ (that would again give $\det a = 1$) may become singular, as $[k] = 0$ for q given by (1.4). From this point of view, the convention $\det a = 1$ is neither convenient nor always possible.

1.3. Outlook and references

Although the WZNW model was introduced [50] in terms of a multivalued action, its solution was first given in the axiomatic approach to conformal current algebra models [41, 47]. The canonical (Lagrangean) approach had to await the discovery of the link between the quantum exchange relations and the Yang–Baxter equation [5]. It was initiated for the WZNW model in [14] and was given a strong impetus by [27]. Among early subsequent works [8, 6, 35, 17, 18, 29, 7, 33, 34] we would like to single out the development by Gawędzki and co-workers [35, 29, 36] of a truly canonical first-order formalism adapted to the problem. The present paper is devoted to a self-contained study of the finite-dimensional zero modes' problem (without recurrent appeal to its infinite-dimensional origin). This problem was first singled out in [1] followed by [4, 15, 30]—among others. It has an interest of its own, exhibiting in a nutshell a number of properties that attract the attention of both physicists and mathematicians: Poisson–Lie symmetry [45, 4, 9, 10, 2, 11], r- (R-) matrices (classical and quantum) [13, 45, 28], dynamical r- (R-) matrices [37, 31, 26, 43, 42, 25, 24, 3]. The study of the $SU(2)$ case in [33] was extended to $SU(n)$ in [38, 32], $s\ell(n)$ being singled out among other simple Lie algebras by the fact that the corresponding quantum R-matrices satisfy quadratic (Hecke algebra) relations. The gauge freedom in the very definition of the zero mode phase space was discussed in [34] and its BRS (co)homology was studied in [22, 23] (for a concise review see [21]). The presence of such a freedom allows us, in particular, to avoid the complications of the unitary gauge advocated in [9, 10].

1.4. Outline of the paper

After sketching (in section 2.1) the derivation of expression (2.10) for Ω_q that generalizes (1.14) to any compact semisimple Lie group G, we study in section 2.2 an extension $\mathcal{M}^{ex}_q$ of the phase space $\mathcal{M}_q$ for $G = SU(n)$ for which one derives a more manageable symplectic form Ω_q^{ex}. In section 2.3 we display the undeformed limit $k \to \infty$ ($q \to 1$) in which the WZ term disappears. The resulting form Ω_1 can be easily inverted. We also display the Hamiltonian vector fields corresponding to the constraints $\chi := \log \frac{D}{\mathcal{D}_q(p)}$ and $P := \frac{1}{n}\sum_{s=1}^n p_s$. In particular,

$$\mathrm{i}\frac{\hat{\partial}}{\partial P}\Omega_q^{ex} = \mathrm{i}\sum_{s=1}^{n}\frac{\hat{\partial}}{\partial p_s}\Omega_q^{ex} = \mathrm{d}\chi = \frac{\mathrm{d}D}{D} - \frac{\mathrm{d}\mathcal{D}_q(p)}{\mathcal{D}_q(p)}. \tag{1.18}$$

Here $\hat{X}\Omega$ means the contraction of the vector field X with the form Ω; we have, e.g.,

$$\frac{\hat{\partial}}{\partial p_s}\,\mathrm{d}p_j = \delta_j^s - \mathrm{d}p_j\frac{\hat{\partial}}{\partial p_s}. \tag{1.19}$$

It is important that these 'momentum maps' remain valid after q-deformation (i.e. for finite k). Section 3 is devoted to inverting the form Ω_q^{ex} (and Ω_q), thus computing PB among zero modes. In section 4.1 we demonstrate that the quasiclassical limit ($k \gg n$, $p_{j\ell} \gg 1$, $\frac{p_{j\ell}}{k}$ finite) of the quantum exchange relations of [38, 32, 39] reproduces the PB relations of section 3. In the rest of section 4 we review the $U_q(s\ell_n)$ symmetry of the quantum matrix algebra and its operator realization.

2. Zero modes' phase space from chiral WZNW 2-form

2.1. From a 2D canonical 3-form to the zero modes' symplectic form

The canonical approach to a field theory in D-dimensional spacetime formulated in [35] (where its sources are cited and reviewed) starts with a closed $(D+1)$-form ω ($=\mathrm{d}\mathbf{L}(x)$ if a Lagrangian D-form $\mathbf{L}(x)$ exists). It allows us to read off the equations of motion while the integral over a $(D-1)$-dimensional space-like surface provides the symplectic form of the theory. A form of this type, called symplectic density, was recently (partly rediscovered and) applied to Yang–Mills, general relativity, Chern–Simons and supergravity theories [40]. In the case of the WZNW model, the 3-form ω can be written as the sum of an exact form and the canonical invariant closed 3-form on the group G,

$$\omega = \mathrm{d}\left\{\frac{1}{2}\,\mathrm{tr}\left(\mathrm{i}g^{-1}\,\mathrm{d}g + \frac{\pi}{k}\mathbf{J}\right){}^*\mathbf{J}\right\} + \frac{k}{12\pi}\,\mathrm{tr}(g^{-1}\,\mathrm{d}g)^3 \tag{2.1}$$

where $\mathbf{J}$ is the current 1-form and ${}^*\mathbf{J}$ is its Hodge dual:

$$\mathbf{J}(x) = j_\mu(x)\,\mathrm{d}x^\mu \qquad {}^*\mathbf{J}(x) = \varepsilon_{\mu\nu}j^\mu(x)\,\mathrm{d}x^\nu \qquad (\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}, \varepsilon_{01} = 1 = \varepsilon^{10}). \tag{2.2}$$

We shall sum up without derivation the implications of equation (2.1).

The equations of motion, obtained as the pull-back of the contractions of ω with the vertical vector fields $\frac{\delta}{\delta j_\mu(x)}$ and $g(x)X\frac{\delta}{\delta g(x)}$, read

$$\mathbf{J} = \frac{k}{2\pi\mathrm{i}}g^{-1}\,\mathrm{d}g \qquad \mathrm{d}\mathbf{J} + \frac{2\pi\mathrm{i}}{k}\mathbf{J}^2 = 0 \quad\Rightarrow\quad \mathrm{d}(\mathbf{J} + {}^*\mathbf{J}) = 0. \tag{2.3}$$

They imply the existence of left and right (Nöther) currents depending on a single light cone variable,

$$j_R = \tfrac{1}{2}(j^0 + j^1) \qquad j_L = \tfrac{1}{2}g(j^1 - j^0)g^{-1} \qquad \partial_+ j_R = 0 = \partial_- j_L$$
$$\text{for}\quad \partial_\pm = \tfrac{1}{2}(\partial_1 \pm \partial_0) \tag{2.4}$$

and the factorization (1.1) of $g(x^0, x^1)$.

The symplectic form $\Omega^{(2)}$ can be expressed in terms of either of the two chiral currents:

$$\begin{aligned}\Omega^{(2)} &= \int_{-\pi}^{\pi} \omega\, \mathrm{d}x^1 \\ &= -\int_{-\pi}^{\pi} \mathrm{d}x\, \mathrm{tr}\left(\mathrm{i}\, \mathrm{d}(j_L\, \mathrm{d}g g^{-1}) + \frac{k}{4\pi}\, \mathrm{d}g g^{-1} (\mathrm{d}g g^{-1})'\right) \\ &= \int_{-\pi}^{\pi} \mathrm{d}x\, \mathrm{tr}\left(\mathrm{i}\, \mathrm{d}(j_R g^{-1}\, \mathrm{d}g) + \frac{k}{4\pi} g^{-1}\, \mathrm{d}g (g^{-1}\, \mathrm{d}g)'\right).\end{aligned} \tag{2.5}$$

Inserting the factorized expression (1.1) for g in (2.5), one can split $\Omega^{(2)}$ into chiral symplectic forms

$$\Omega^{(2)} = \Omega(g_L, M) - \Omega(g_R, M) \tag{2.6}$$

where

$$\Omega(g_C, M) = \frac{k}{4\pi}\left\{\mathrm{tr}\left(\int_{-\pi}^{\pi} \mathrm{d}x \big(g_C^{-1}\, \mathrm{d}g_C \big(g_C^{-1}\, \mathrm{d}g_C\big)'\big) + b_C^{-1}\, \mathrm{d}b_C\, \mathrm{d}M M^{-1}\right) - \rho(M)\right\} \tag{2.7}$$

with

$$b_C := g_C(-\pi) \qquad M = b_C^{-1} g_C(\pi) \quad \big(= g_C^{-1}(x) g_C(x + 2\pi)\big) \qquad C = L, R. \tag{2.8}$$

The cumbersome (ill defined) WZ term $\rho(M)$ (satisfying (1.15)) has been added and subtracted from the two chiral terms to ensure $\mathrm{d}\Omega = 0$. An alternative approach, introducing quasi-Poisson manifolds [2] (for which the Jacobi identity satisfied by proper PB is replaced by a weaker condition) is developed in [11].

Finally, substituting $g_L(x)$ by its expression (1.3), we find

$$\Omega(g_L, M) = \Omega(u, M_p) + \omega_q(M_p) + \Omega_q(a, M_p) \tag{2.9}$$

where

$$\Omega_q(a, M_p) = \frac{k}{4\pi}\left\{\mathrm{tr}\left(\mathrm{d}a a^{-1}\big(2\, \mathrm{d}M_p M_p^{-1} + M_p\, \mathrm{d}a a^{-1} M_p^{-1}\big)\right) - \rho(a^{-1} M_p a)\right\} - \omega_q(M_p) \tag{2.10}$$

and ω_q is an arbitrary closed 2-form (which will be restricted further by some symmetry conditions). For $G = SU(2)$ there is a single variable p, hence $\omega_q(M_p) \equiv 0$ and (2.10) coincides with (1.14).

A detailed derivation of the results formulated in this subsection will be presented elsewhere.

2.2. Basis of right invariant 1-forms: An extended phase space and a privileged choice of ω_q for $G = SU(n)$

We shall now write down the first two terms in expression (2.10) as sums of products of right invariant forms. To this end we shall use the Cartan–Weyl basis $\{h_i, e_\alpha\}$, α running through the positive roots of $\mathcal{G}_{\mathbb{C}}$ (in its n-dimensional fundamental representation) satisfying

$$[h_i, h_j] = 0 \qquad [h_i, e_{\pm\alpha}] = \pm 2\frac{(\alpha|\alpha_j)}{|\alpha_j|^2} e_{\pm\alpha} \qquad [e_{\alpha_i}, e_{-\alpha_j}] = \delta_{ij} h_j \tag{2.11}$$

$(i, j = 1, \ldots, r := \mathrm{rank}\, \mathcal{G})$, and shall write

$$\hat{p} = \sum_{j=1}^{r} p_{\alpha_j} h^j \qquad \text{with} \quad \mathrm{tr}(h^i h_j) = \delta^i_j \quad (\text{and } \mathrm{tr}(e_\alpha e_{-\beta}) = \delta_{\alpha\beta}) \tag{2.12}$$

(thus $\{h_i\}$ and $\{h^j\}$ define dual bases of diagonal matrices). Let further Θ^j, $\Theta^{\pm\alpha}$ and $\frac{\mathrm{d}D}{D}$ be the corresponding right invariant 1-forms in $T^*G^{ex}_{\mathbb{C}}$,

$$G^{ex}_{\mathbb{C}} := (G \times \mathbb{R}_+)_{\mathbb{C}} \tag{2.13}$$

defined by

$$\Theta^j = -\mathrm{i}\,\mathrm{tr}(a^{-1}h^j\,\mathrm{d}a) \qquad \Theta^{\pm\alpha} = -\mathrm{i}\,\mathrm{tr}(a^{-1}e_{\mp\alpha}\,\mathrm{d}a) \qquad \frac{\mathrm{d}D}{D} = \mathrm{tr}(\mathrm{d}a a^{-1}). \tag{2.14}$$

It then follows that

$$-\mathrm{i}\,\mathrm{d}aa^{-1} = \sum_{j=1}^{r}\Theta^j h_j + \sum_{\alpha>0}(\Theta^\alpha e_\alpha + \Theta^{-\alpha}e_{-a}) - \frac{\mathrm{i}}{n}\frac{\mathrm{d}D}{D}\mathbb{1} \tag{2.15}$$

where $D = \det a > 0$ and $\mathbb{1}$ is the $n \times n$ unit matrix.

If G is compact, then the forms Θ^j are real while $\Theta^{-\alpha}$ are complex conjugate to Θ^α. We also note that the (Lie algebra valued) 1-form (2.15) is not closed but defines a flat connection, the Θs satisfying the Cartan–Maurer relations. We shall use, in particular,

$$\mathrm{d}\Theta^j = \mathrm{i}\sum_{\alpha>0}(\Lambda^j|\alpha)\Theta^\alpha\Theta^{-\alpha} \qquad \big((\Lambda^j|\alpha_\ell) = \delta^j_\ell\big) \tag{2.16}$$

Λ^j being the *fundamental weights* of $\mathcal{G}$.

Inserting (2.12) into the first term on the right-hand side of (2.10) and using (1.15) and (2.14), we deduce

$$\frac{k}{2\pi}\,\mathrm{tr}\big(\mathrm{d}aa^{-1}\,\mathrm{d}M_pM_p^{-1}\big) = \mathrm{i}\,\mathrm{tr}(\mathrm{d}aa^{-1}\hat{p}) = \sum_{j=1}^{r}\mathrm{d}p_{\alpha_j}\Theta^j. \tag{2.17}$$

The second term is expressed as a sum of products of conjugate off-diagonal forms:

$$\frac{k}{4\pi}\,\mathrm{tr}\big(\mathrm{d}aa^{-1}M_p\,\mathrm{d}aa^{-1}M_p^{-1}\big) = \frac{k}{4\pi}(\overline{q} - q)\sum_{\alpha>0}[2p_\alpha]\Theta^\alpha\Theta^{-\alpha} \tag{2.18}$$

where p_α is a linear functional on the roots:

$$p_\alpha = \sum_{j=1}^{r}(\Lambda^j|\alpha)p_{\alpha_j} \qquad \text{for} \quad \alpha = \sum_{j=1}^{r}(\Lambda^j|\alpha)\alpha_j. \tag{2.19}$$

Here $(\Lambda^j|\alpha) \in \mathbb{Z}_+$ and we have the relation

$$Ad_{M_p}e_\alpha := M_pe_\alpha M_p^{-1} = \overline{q}^{2p_\alpha}e_\alpha. \tag{2.20}$$

At this point we shall specialize to the case $G = SU(n)$ and will view the $(n-1)(n+2)$-dimensional symplectic manifold $\mathcal{M}_q = \mathcal{M}_q(n)$ as a submanifold of codimension 2 in the *extended* ($n(n+1)$-dimensional) *phase space* $\mathcal{M}^{ex}_q$ spanned by p_i and a^j_α ($i, j, \alpha = 1, \ldots, n$) regarded as independent variables:

$$\mathcal{M}_q = \left\{\big(p_i, a^j_\alpha\big) \in \mathcal{M}^{ex}_q;\ P := \frac{1}{n}\sum_{s=1}^{n}p_s = 0,\ \chi := \log\frac{D}{\mathcal{D}_q(p)} = 0\right\}. \tag{2.21}$$

We introduce the Weyl basis $\big\{e^j_i\big\}$ of $n \times n$ matrices satisfying

$$e^j_ie^\ell_k = \delta^j_ke^\ell_i \qquad \big(e^j_i\big)^\ell_k = \delta^\ell_i\delta^j_k \qquad i, j, k, \ell = 1, \ldots, n. \tag{2.22}$$

The positive roots α_{ij} $(i < j)$ of $su(n)$ correspond to raising operators, e_i^j, while $-\alpha_{ij}$ are associated with lowering operators, e_j^i. Equation (2.15) now assumes a simple explicit form:

$$-\mathrm{i}\,\mathrm{d}aa^{-1} = \Theta_k^j e_j^k \qquad \left(\equiv \sum_{j,k=1}^{n} \Theta_k^j e_j^k\right) \qquad \Theta_k^j = -\mathrm{i}\,\mathrm{tr}\big(e_k^j\,\mathrm{d}aa^{-1}\big) = -\mathrm{i}\,\mathrm{d}a_\sigma^j (a^{-1})_k^\sigma. \tag{2.23}$$

The general Cartan–Maurer relations (which incorporate (2.16)) are written simply as

$$\mathrm{d}\Theta_k^j = \mathrm{i}\Theta_s^j\Theta_k^s. \tag{2.24}$$

Recalling that relation (1.7) is invariant under simultaneous permutation of the rows of the matrix $\big(a_\alpha^j\big)$ and of p_j (i.e. under the action on both sides of the $su(n)$ Weyl group), we shall also require permutation invariance of the extended form $\omega_q^{ex}(p)$. We shall determine $\omega_q(M_p) = \omega_q^{ex}(p)|_{P=0}$ by further demanding that the symplectic form Ω_q^{ex} on $\mathcal{M}_q^{ex}$,

$$\Omega_q^{ex} = \sum_{s=1}^{n} \mathrm{d}p_s\Theta_s^s - \frac{k}{4\pi}\left\{(q-\overline{q})\sum_{j<\ell}[2p_{j\ell}]\Theta_\ell^j\Theta_j^\ell + \rho(a^{-1}M_p a)\right\} - \omega_q^{ex}(p) \tag{2.25}$$

will reduce to Ω_q on the surface $\mathcal{M}_q \subset \mathcal{M}_q^{ex}$. In order to implement this last condition, we shall require that the terms involving $\mathrm{d}P$ cancel in the difference

$$-\omega_q(M_p) = \left(-\mathrm{i}\,\mathrm{d}P\frac{\mathrm{d}\mathcal{D}_q(p)}{\mathcal{D}_q(p)}\right) - \omega_q^{ex}(p). \tag{2.26}$$

Inserting the expression (cf (1.7)) for $\mathcal{D}_q(p)$ which implies

$$\frac{\mathrm{d}\mathcal{D}_q(p)}{\mathcal{D}_q(p)} = \frac{\pi}{k}\sum_{j<\ell}\cot\left(\frac{\pi}{k}p_{j\ell}\right)\mathrm{d}p_{j\ell} \tag{2.27}$$

we find a form $\omega_q^{ex}(p)$ satisfying all the above conditions:

$$\omega_q^{ex}(p) = \mathrm{i}\frac{\pi}{k}\sum_{j<\ell}\cot\left(\frac{\pi}{k}p_{j\ell}\right)\mathrm{d}p_j\,\mathrm{d}p_\ell. \tag{2.28}$$

Indeed, using the relation

$$p_j = P + \frac{1}{n}\sum_{s=1}^{n} p_{js} \tag{2.29}$$

we deduce

$$\omega_q(M_p) = \frac{\mathrm{i}\pi}{nk}\sum_{1\leqslant j<\ell<m\leqslant n}\left(\cot\left(\frac{\pi}{k}p_{j\ell}\right) + \cot\left(\frac{\pi}{k}p_{\ell m}\right) - \cot\left(\frac{\pi}{k}p_{jm}\right)\right)\mathrm{d}p_{j\ell}\,\mathrm{d}p_{\ell m} \tag{2.30}$$

(note that for $n = 2$ there is no triple j, ℓ, m satisfying the above inequalities so that the form $\omega_q(M_p)$ vanishes, as it should, while $\omega_q^{ex}(p)$ (2.28) reduces to a single term: $\omega_q^{ex}(p) = \mathrm{i}\frac{\pi}{k}\cot\big(\frac{\pi}{k}p_{12}\big)\,\mathrm{d}p_1\,\mathrm{d}p_2$).

We observe the relative simplicity of the extended symplectic form (2.25), (2.28) as compared with Ω_q (obtained from (2.10) by inserting (2.17) with

$$p_{\alpha_j} = p_{jj+1} \qquad \Theta^j = \frac{n-j}{n}\sum_{s=1}^{j}\Theta_s^s - \frac{j}{n}\sum_{s=j+1}^{n}\Theta_s^s \tag{2.31}$$

(2.18) and (2.30)). It is, therefore, rewarding to know that the PB we are interested in can be computed using the simpler expression Ω_q^{ex}, as we shall see in section 3. In the next subsection we shall display this property for the $k \to \infty$ limit theory.

2.3. Right invariant vector fields: the limit $k \to \infty$. Dirac brackets

It is easy to display the basis of right invariant vector fields $\left\{\frac{\partial}{\partial p_\ell}, V_j^k\right\}$ dual to the basis $\left\{\mathrm{d}p_\ell, \Theta_k^j\right\}$ of 1-forms:

$$V_j^k = \mathrm{i}\,\mathrm{tr}\left(e_j^k a \frac{\partial}{\partial a}\right) = \mathrm{i}a_\sigma^k \frac{\partial}{\partial a_\sigma^j}. \tag{2.32}$$

Indeed, contracting the form Θ_m^ℓ (2.23) with V_j^k, we find

$$\hat{V}_j^k \Theta_m^\ell = \mathrm{tr}\left(e_j^k a a^{-1} e_m^\ell\right) = \delta_j^\ell \delta_m^k \qquad \hat{V}_j^k\, \mathrm{d}p_\ell = 0. \tag{2.33}$$

Obviously,

$$\frac{\hat{\partial}}{\partial p_j}\Theta_m^\ell = 0 \qquad \frac{\hat{\partial}}{\partial p_j}\,\mathrm{d}p_\ell = \delta_\ell^j.$$

This would allow us to invert the form Ω_q^{ex} but for the WZ term.

We shall profit from the above remark taking the limit $k \to \infty$ in which the WZ term disappears. Indeed, using the expression for q in (1.4), we find

$$\lim_{k\to\infty} \frac{k}{2\pi}(\overline{q} - q) = \mathrm{i} \qquad \frac{1}{2}\lim_{k\to\infty}[2p] = p \tag{2.34}$$

and hence,

$$\begin{aligned}\Omega_1^{ex}(a, p) &= \sum_{s=1}^n \mathrm{d}p_s \Theta_s^s + \mathrm{i}\sum_{1\leqslant j<\ell\leqslant n} p_{j\ell}\Theta_\ell^j \Theta_j^\ell - \omega_1(p) \\ &= \mathrm{d}\sum_{s=1}^n p_s\Theta_s^s - \mathrm{i}\sum_{1\leqslant j<\ell\leqslant n}\frac{\mathrm{d}p_j\,\mathrm{d}p_\ell}{p_{j\ell}}.\end{aligned} \tag{2.35}$$

Here we have set

$$\lim_{k\to\infty}\frac{k}{4\pi}\rho(a^{-1}M_p a) = 0. \tag{2.36}$$

In fact, since the right-hand side of (2.35) is a closed 2-form, it follows that

$$\lim_{k\to\infty}\frac{k}{4\pi}\,\mathrm{tr}(\mathrm{d}MM^{-1})^3 = 0 \qquad \text{for} \quad M = a^{-1}M_p a. \tag{2.37}$$

We conclude that $\frac{k}{4\pi}\rho$ can also be chosen to vanish in this limit—a property that can be derived from the expression for $\rho(a^{-1}M_p a)$ given in section 3.

As anticipated, it is straightforward to invert the 2-form (2.35). The result can be encoded in the Poisson bivector

$$\mathcal{P} = \sum_{s=1}^n V_s^s \wedge \frac{\partial}{\partial p_s} + \mathrm{i}\sum_{1\leqslant j<\ell\leqslant n}\frac{1}{p_{j\ell}}\left(V_j^\ell \wedge V_\ell^j - V_j^j \wedge V_\ell^\ell\right) \tag{2.38}$$

which gives rise to the following PB:

$$\{p_j, p_\ell\} = 0 \qquad \left\{a_\alpha^j, p_\ell\right\} = \mathrm{i}\delta_\ell^j a_\alpha^j \tag{2.39}$$

and

$$\left\{a_\alpha^j, a_\beta^\ell\right\} = r^{(1)}(p)_{j'\ell'}^{j\ell} a_\alpha^{j'} a_\beta^{\ell'} \qquad \left(\text{i.e. } \{a_1, a_2\} = r_{12}^{(1)}(p) a_1 a_2\right) \tag{2.40}$$

where the undeformed classical dynamical r-matrix is given by

$$r^{(1)}(p)_{j'\ell'}^{j\ell} = \begin{cases}\frac{\mathrm{i}}{p_{j\ell}}\left(\delta_{j'}^j\delta_{\ell'}^\ell - \delta_{\ell'}^j\delta_{j'}^\ell\right) & \text{for} \quad j \neq \ell \\ 0 & \text{for} \quad j = \ell\end{cases}. \tag{2.41}$$

For a general Poisson manifold $\mathcal{M}$ with a pair of second class constraints P and χ the Dirac brackets $\{f, g\}_D$ [20] of two arbitrary functions on $\mathcal{M}$ are expressed in terms of their PB as

$$\{f, g\}_D = \{f, g\} + \frac{1}{\{P, \chi\}}\,(\{f, P\}\{\chi, g\} - \{f, \chi\}\{P, g\})\,. \tag{2.42}$$

We shall verify that in the case at hand

$$\{p_{j\ell}, P\} = 0 = \{p_{j\ell}, \chi\} \qquad \left\{a_\alpha^j, \chi\right\} = 0. \tag{2.43}$$

The first pair of equations implies that $p_{j\ell}$ are 'observables' on $\mathcal{M}_q \subset \mathcal{M}_q^{ex}$, so that $\{p_{j\ell}, f\}_D = \{p_{j\ell}, f\}$ for any function f on $\mathcal{M}_q^{ex}$; in particular,

$$\left\{p_{j\ell}, a_\alpha^m\right\} = \mathrm{i}\big(\delta_\ell^m - \delta_j^m\big)a_\alpha^m = \left\{p_{j\ell}, a_\alpha^m\right\}_D. \tag{2.44}$$

The last equation (2.43) is sufficient to assert that the PB (2.40) coincide with the corresponding Dirac brackets.

Although it is easy to verify (2.43) directly, using (2.38)–(2.40), we shall give a more general derivation that will apply to the case of finite k ($q \neq 1$) as well. To this end we shall use the momentum maps

$$\mathrm{i}\frac{\hat{\partial}}{\partial P}\Omega_1 = \mathrm{i}\sum_{s=1}^{n}\frac{\hat{\partial}}{\partial p_s}\Omega_1 = \mathrm{i}\sum_{s=1}^{n}\Theta_s^s - \sum_{1\leqslant j<\ell\leqslant n}\frac{\mathrm{d}p_{j\ell}}{p_{j\ell}} = \mathrm{d}\chi \qquad -\frac{1}{n}\sum_{s=1}^{n}\hat{V}_s^s\Omega_1 = \mathrm{d}P. \tag{2.45}$$

Displaying the Hamiltonian vector fields corresponding to χ and P, equation (2.45) allows us to compute any PB of the constraints; in particular,

$$\begin{aligned} &\left\{\chi, a_\alpha^j\right\} = \mathrm{i}\frac{\hat{\partial}}{\partial P}\,\mathrm{d}a_\alpha^j = 0 && \{\chi, p_{j\ell}\} = \mathrm{i}\frac{\hat{\partial}}{\partial P}\,\mathrm{d}p_{j\ell} = 0 \\ &\{p_{j\ell}, P\} = \frac{1}{n}\sum_{s=1}^{n}\hat{V}_s^s\,\mathrm{d}p_{j\ell} = 0 && \{P, \chi\} = -\mathrm{i}. \end{aligned} \tag{2.46}$$

We find, on the other hand,

$$\left\{a_\alpha^j, p_\ell\right\}_D = \left\{a_\alpha^j, p_\ell\right\} + \mathrm{i}\left\{a_\alpha^j, P\right\}\{\chi, p_\ell\} = \mathrm{i}a_\alpha^j\left(\delta_\ell^j - \frac{1}{n}\right). \tag{2.47}$$

3. Inverting Ω_q^{ex}: PB in $\mathcal{M}_q(n)$

3.1. The WZ form

It was Gawędzki [35] (see also [29]) who introduced in the early 1990s the WZ 2-form $\rho(M)$ and described its relation to the non-degenerate (constant) solutions of the classical Yang–Baxter equation (CYBE). Gradually, a more general and complete understanding of such a relation has been worked out [10, 30]. We shall only deal here with a special case of the outcome of [30] corresponding essentially to the early discussion in [29].

We shall again start with an arbitrary semisimple matrix Lie group G with Lie algebra $\mathcal{G}$. For an arbitrary pair $\{t^a\}$, $\{T_b\}$ of dual bases in $\mathcal{G}$, we can write the Killing metric tensor η_{ab} and its inverse, η^{ab}, as

$$\eta_{ab} = \mathrm{tr}(T_a T_b) \qquad \eta^{ab} = \mathrm{tr}(t^a t^b) \qquad \text{for} \quad \mathrm{tr}(t^a T_b) = \delta_b^a. \tag{3.1}$$

In the Cartan–Weyl basis $\{T_a\} = \{h_i, e_{\pm\alpha}\}$ we have $\{t^a = h^i, e_{\mp\alpha}\}$ and the nonzero elements of η are

$$\eta_{ij} = \mathrm{tr}\, h_i h_j = (\alpha_i|\alpha_j) \qquad \eta_{\alpha\beta} = \mathrm{tr}\, e_\alpha e_\beta = \delta_{\alpha,-\beta} \tag{3.2}$$

(where the norm square of the highest root is fixed to 2).

The polarized Casimir invariant $C_{12} \in \mathrm{Sym}\,(\mathcal{G} \otimes \mathcal{G})$, given (in Faddeev's notation [28]) by

$$C_{12} = \eta_{ab} t_1^a t_2^b \qquad (= T_a \otimes t^a \equiv t^a \otimes T_a) = h_{i1} h_2^i + \sum_\alpha e_{\alpha 1} e_{-\alpha 2} \tag{3.3}$$

where the sum is taken over all, positive and negative, roots α, plays the role of the unit operator on $\mathcal{G}$:

$$CX := \mathrm{tr}_2(C_{12} X_2) = X \quad (\equiv X_1) \qquad \text{for} \quad X \in \mathcal{G}. \tag{3.4}$$

Let $r_{12} = -r_{21}$ $(\in \mathcal{G} \wedge \mathcal{G})$ be a solution of the *modified* CYBE

$$[r_{12}, r_{13} + r_{23}] + [r_{13}, r_{23}] = [C_{12}, C_{23}] \quad (= -f_{abc} t^a t^b t^c) \tag{3.5}$$

and let r be the corresponding operator ($r : \mathcal{G} \to \mathcal{G}$) defined by taking the trace in the second argument as in (3.4):

$$rX := \mathrm{tr}_2(r_{12} X_2) \qquad \text{for} \quad X \in \mathcal{G} \quad \Rightarrow \quad r_{12} = rC_{12}. \tag{3.6}$$

Proposition 3.1. *Let the 2-form $\rho(M)$ be written in terms of a skew-symmetric kernel $K(M)_{12} \in \mathcal{G} \wedge \mathcal{G}$ for $M \in G_0$ where G_0 is an open dense neighbourhood of the group unit in which the operator $(1 - Ad_M)r + 1 + Ad_M$, $Ad_M X := MXM^{-1}$, is invertible, and let $K(M)$ be the corresponding operator $K(M) : \mathcal{G} \to \mathcal{G}$ defined as in (3.6),*

$$\rho(M) = \tfrac{1}{2}\,\mathrm{tr}(\mathrm{d}MM^{-1} K(M)\,\mathrm{d}MM^{-1}) \qquad (\mathrm{tr}(XK(M)X) = 0 \ \forall X \in \mathcal{G}). \tag{3.7}$$

Assume further that

$$K(M) = ((1 + Ad_M)r + 1 - Ad_M)((1 - Ad_M)\,r + 1 + Ad_M)^{-1} \tag{3.8}$$

so that $K(1) = r$. Then $\rho(M)$ satisfies

$$\mathrm{d}\rho(M) = \tfrac{1}{3}\,\mathrm{tr}(\mathrm{d}MM^{-1})^3 \tag{3.9}$$

iff r_{12} (related to r by (3.6)) satisfies the modified CYBE (3.5).

The statement is a corollary of propositions 1 and 2 of [30]; see also the earlier discussion in [29].

Remark 3.1. The modified CYBE (3.5) for r_{12} is equivalent to the standard CYBE

$$\left[r_{12}^\pm, r_{13}^\pm + r_{23}^\pm\right] + \left[r_{13}^\pm, r_{23}^\pm\right] = 0 \qquad \text{for} \quad r_{12}^\pm = r_{12} \pm C_{12}. \tag{3.10}$$

In fact, [29] deals with equation (3.10).

Remark 3.2. Using (3.8), it is easy to check that the skewsymmetry of $K(M)$ is equivalent to that of r, ${}^tK(M) = -K(M) \Leftrightarrow {}^tr = -r$ where the transposition t is with respect to the invariant bilinear form tr.

Remark 3.3. One can consider a more general ansatz of type (3.8) allowing the operator r to depend on M. Then one has to deal with a 'dynamical' version of the (modified) CYBE including differentiation in the group parameters—see Eqs. (1.3) and (3.8) of [30]. One argues in [10] that a constant classical r-matrix cannot correspond to a compact group G. It is well known, indeed, that the modified CYBE (3.5) has no real solution in $\mathcal{G} \wedge \mathcal{G}$ for $\mathcal{G}$ compact. We shall however stick to the above simple choice which uses a complex 2-form ρ. As noted in the introduction, the use of the simple constant r-matrix for the PB of the zero modes a_α^j is perfectly admissible because the freedom in their choice does not affect the properties of $u_j^A(x)$ which always transform covariantly under left shifts of the compact group G.

Using (3.7) and (3.8), we can present the WZ 2-form in (2.10) as

$$\rho(a^{-1}M_p a) = \tfrac{1}{2}\,\mathrm{tr}\{(\mathrm{d}M_p M_p^{-1} - A_-(\mathrm{d}a a^{-1}))K^a(\mathrm{d}M_p M_p^{-1} - A_-(\mathrm{d}a a^{-1}))\}. \tag{3.11}$$

Here and below we are using the operators

$$A_\pm := 1 \pm Ad_{M_p} \qquad A_-\mathrm{d}M_p M_p^{-1} = 0 = (A_+ - 2)\,\mathrm{d}M_p M_p^{-1} \tag{3.12}$$

while K^a is given by

$$\begin{aligned} K^a &:= Ad_a K(a^{-1}M_p a) Ad_a^{-1} = (A_+ r^a + A_-)(A_- r^a + A_+)^{-1} \\ r^a &:= Ad_a r Ad_a^{-1} \end{aligned} \tag{3.13}$$

(K^a and r^a are skewsymmetric together with $K(M)$ and r).

We note that $\rho(a^{-1}M_p a)$ coincides with its extension on $\mathcal{M}_q^{ex}(n)$. This is obviously true for the 3-form $\mathrm{d}\rho(M)$ (3.9) for $M = a^{-1}M_p a$. Indeed, the contribution of the term proportional to $\mathrm{d}P$ is given by

$$\mathrm{d}\rho^{ex}(a^{-1}M_p a) - \mathrm{d}\rho(a^{-1}M_p a) = \frac{2\pi\mathrm{i}}{k}\,\mathrm{d}P\,\mathrm{tr}(\mathrm{d}M M^{-1})^2 \equiv 0. \tag{3.14}$$

(Remember that the diagonal monodromy M_p enters Ω_q^{ex} through

$$\mathrm{d}M_p M_p^{-1} = \frac{2\pi\mathrm{i}}{k}\sum_{s=1}^{n}\mathrm{d}p_s e_s^s \equiv \frac{2\pi\mathrm{i}}{k}\,(\mathrm{d}P\,\mathbb{1} + \mathrm{d}\hat{p}) \tag{3.15}$$

cf (2.12), and $\det a$ is *not* set equal to $\mathcal{D}_q(p)$.) Since ρ is *defined* by equation (3.9), it can be left unchanged in the extended phase space. This is certainly true for expressions (3.11)–(3.13) provided we take—as we will—the *standard solution* of the modified CYBE (3.5)

$$r_{12} = \sum_{\alpha>0}(e_\alpha \otimes e_{-\alpha} - e_{-\alpha}\otimes e_\alpha) \equiv \sum_{\alpha>0}\left(e_{1\alpha}e_2^\alpha - e_1^\alpha e_{2\alpha}\right) \qquad (e^\alpha := e_{-\alpha}) \tag{3.16}$$

for which

$$r e_{\pm\alpha} = \pm e_{\pm\alpha} \qquad \text{for} \quad \alpha > 0 \qquad r h_i = 0 = r\mathbb{1}. \tag{3.17}$$

3.2. Poisson bivector for Ω_q^{ex}

We shall first establish relation (1.18) which, according to (2.46), is sufficient to prove that the PB of a_α^j can be computed using the form Ω_q^{ex} (2.25), (2.28) on $\mathcal{M}_q^{ex}(n)$. Equation (1.18) follows from (2.26)–(2.28) and (3.14) (together with the subsequent argument), which imply

$$\frac{\hat{\partial}}{\partial P}\rho(a^{-1}M_p a) = 0. \tag{3.18}$$

Similarly, one can deduce

$$\sum_{s=1}^{n}\hat{V}_s^s\rho(a^{-1}M_p a) = 0 \quad \Rightarrow \quad -\frac{1}{n}\sum_{s=1}^{n}\hat{V}_s^s\Omega_q^{ex} = \mathrm{d}P \tag{3.19}$$

which extends the second equation (2.45) to $q \neq 1$.

Using (3.11)–(3.13), we can write the extended symplectic form (2.25) as

$$\begin{aligned} \Omega_q^{ex} = {} & \sum_{s=1}^{n}\mathrm{d}p_s\Theta_s^s + \frac{k}{2\pi}\sum_{j\neq\ell,r\neq s}\Theta_\ell^j\Theta_s^r[(\omega - X)^{-1}]_{jr}^{\ell s} + \sum_{j,r\neq s,t\neq q}\mathrm{d}p_j\Theta_s^r X_{jq}^{jt}[(\omega - X)^{-1}]_{tr}^{qs} \\ & - \frac{\pi}{2k}\sum_{j\neq\ell}\mathrm{d}p_j\,\mathrm{d}p_\ell\left(\omega^{jl} + X_{j\ell}^{j\ell} + \sum_{s\neq t,s'\neq t'}X_{jt}^{js}[(\omega - X)^{-1}]_{ss'}^{tt'}X_{t'\ell}^{s'\ell}\right) \end{aligned} \tag{3.20}$$

where

$$\omega^{j\ell} := \mathrm{i}\cot\frac{\pi}{k}p_{j\ell} \qquad \omega^{nj}_{m\ell} = -\omega^{j\ell}\delta^j_m\,\delta^n_\ell = \omega^{nj}\delta^n_\ell\delta^j_m \tag{3.21}$$

so that

$$\omega^{ex}_q(p) = \frac{\pi}{k}\sum_{j\neq\ell}\omega^{j\ell}\mathrm{d}p_j\,\mathrm{d}p_\ell \qquad \frac{A_+}{A_-}e^j_\ell = -\omega^{j\ell}\,e^j_\ell \equiv \omega^{nj}_{m\ell}\,e^m_n \tag{3.22}$$

(because $Ad_{M_p}e^j_\ell = q^{2p_{j\ell}}e^j_\ell$) and $X^{nj}_{m\ell}$ is defined as

$$r^a e^j_\ell = -X^{nj}_{m\ell}e^m_n \quad \Rightarrow \quad X_{12} = -Ad_{a_1a_2}r_{12} \quad (X_{12} = -X_{21}). \tag{3.23}$$

To derive (3.21), one uses (2.10), (3.11), (2.23) and (3.15) as well as (3.21), (3.22). This allows us to present Ω^{ex}_q as

$$\Omega^{ex}_q = -\omega^{ex}_q(p) - \frac{k}{8\pi}\,\mathrm{tr}\big\{(A_-\mathrm{d}aa^{-1})(A_+ - K^aA_-)(\mathrm{d}aa^{-1}) \\ + 2\,\mathrm{d}M_pM_p^{-1}(2 - K^aA_-)(\mathrm{d}aa^{-1}) + \mathrm{d}M_pM_p^{-1}K^a\left(\mathrm{d}M_pM_p^{-1}\right)\big\} \tag{3.24}$$

Note that the only nonzero contribution of the diagonal elements of $\mathrm{d}aa^{-1}$ comes through the term

$$-\frac{k}{2\pi}\,\mathrm{tr}(\mathrm{d}M_pM_p^{-1}\,\mathrm{d}aa^{-1}) = \sum_{s=1}^{n}\mathrm{d}p_s\Theta^s_s. \tag{3.25}$$

One also uses relation (3.13) and its corollary

$$(A_+ - K^aA_-)r^a = K^aA_+ - A_- \quad \Rightarrow \quad A_+ - K^aA_- = 4Ad_{M_p}(r^aA_- + A_+)^{-1} \tag{3.26}$$

as well as

$$\frac{k}{4\pi}\,\mathrm{tr}\,\mathrm{d}M_pM_p^{-1}K^aA_-(\mathrm{d}aa^{-1}) = \frac{k}{2\pi}\,\mathrm{tr}\,\mathrm{d}M_pM_p^{-1}r^a\left(r^a + \frac{A_+}{A_-}\right)^{-1}\mathrm{d}aa^{-1} \\ = \sum_{j,r\neq s,t\neq q}\mathrm{d}p_j\Theta^r_sX^{jt}_{jq}[(\omega - X)^{-1}]^{qs}_{tr} \tag{3.27}$$

and

$$\mathrm{tr}\,\mathrm{d}M_pM_p^{-1}K^a\left(\mathrm{d}M_pM_p^{-1}\right) = \mathrm{tr}\,\mathrm{d}M_pM_p^{-1}r^a\left(\mathrm{d}M_pM_p^{-1}\right) \\ + \mathrm{tr}\,\mathrm{d}M_pM_p^{-1}r^a\left[\left(1 + \frac{A_-}{A_+}r^a\right)^{-1} - 1\right]\left(\mathrm{d}M_pM_p^{-1}\right) \tag{3.28}$$

where the second term on the right-hand side gives

$$\mathrm{tr}\big(r^a\,\mathrm{d}M_pM_p^{-1}\big)\left[\left(1 + \frac{A_-}{A_+}r^a\right)^{-1}\frac{A_-}{A_+}r^a\right]\left(\mathrm{d}M_pM_p^{-1}\right) \\ = \frac{4\pi^2}{k^2}\sum_{j\neq\ell}\mathrm{d}p_j\,\mathrm{d}p_\ell\sum_{s\neq t,s'\neq t'}X^{js}_{jt}[(\omega - X)^{-1}]^{tt'}_{ss'}X^{s'\ell}_{t'\ell}. \tag{3.29}$$

The PB derived from Ω^{ex}_q can be compactly written in terms of the Poisson bivector

$$\mathcal{P} = \sum_{m=1}^{n}V^m_m\wedge\frac{\partial}{\partial p_m} + \frac{\pi}{2k}\left(\sum_{n\neq m}\omega^{nm}\left(V^n_m\wedge V^m_n - V^m_m\wedge V^n_n\right) - \sum_{n,m,s,t}X^{ns}_{mt}V^m_n\wedge V^t_s\right) \tag{3.30}$$

obeying the operator equation

$$\mathcal{P}_{12}\left(\Omega^{ex}_q\right)_{23} = I_{13}. \tag{3.31}$$

Here I is the mixed (1, 1)-tensor

$$I = \sum_{j=1}^{n} \left(\frac{\partial}{\partial p_j} \otimes \mathrm{d}p_j + V_j^j \otimes \Theta_j^j \right) + \sum_{j \neq \ell} V_\ell^j \otimes \Theta_j^\ell \tag{3.32}$$

which plays the role of the identity operator in the space of 1-forms Θ, resp. vector fields X in the sense

$$\begin{aligned} \Theta I &:= \sum_{j=1}^{n} \Theta\left(\frac{\partial}{\partial p_j}\right) \mathrm{d}p_j + \sum_{j,\ell=1}^{n} \Theta\big(V_\ell^j\big)\Theta_j^\ell = \Theta \qquad (\Theta(X) \equiv \hat{X}\Theta) \\ I X &:= \sum_{j=1}^{n} \frac{\partial}{\partial p_j} \mathrm{d}p_j(X) + \sum_{j,\ell=1}^{n} V_\ell^j \Theta_j^\ell(X) = X. \end{aligned} \tag{3.33}$$

We find, in particular,

$$\{a_1, a_2\} \equiv \mathcal{P}_{12}(a_1, a_2) = r_{12}(p) a_1 a_2 - \frac{\pi}{k} a_1 a_2 r_{12} \tag{3.34}$$

where

$$r(p)_{j'\ell'}^{j\ell} = \begin{cases} \mathrm{i}\frac{\pi}{k} \cot\big(\frac{\pi}{k} p_{j\ell}\big)\big(\delta_{j'}^j \delta_{\ell'}^\ell - \delta_{\ell'}^j \delta_{j'}^\ell\big) & \text{for} \quad j \neq \ell \\ 0 & \text{for} \quad j = \ell \end{cases} \tag{3.35}$$

and

$$r_{\alpha'\beta'}^{\alpha\beta} = -\epsilon_{\alpha\beta}\, \delta_{\beta'}^\alpha\, \delta_{\alpha'}^\beta \tag{3.36}$$

(cf (3.23) for the standard solution (3.16), (3.17)).

The other two basic PB coincide with those in (2.39) (and the Dirac bracket $\{a_a^j, p_\ell\}$ with (2.47)).

The operators in the triple tensor product $\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n$

$$r_{ab}^{\pm}(p) = r_{ab}(p) \pm \frac{\pi}{k} C_{ab} \qquad a, b = 1, 2, 3 \quad a < b \tag{3.37}$$

satisfy the dynamical CYBE [26]

$$\big[r_{12}^{\pm}(p), r_{13}^{\pm}(p) + r_{23}^{\pm}(p)\big] + \big[r_{13}^{\pm}(p), r_{23}^{\pm}(p)\big] + \mathrm{Alt}(\mathrm{d}r^{\pm}) = 0 \tag{3.38}$$

where

$$\mathrm{Alt}(\mathrm{d}r^{\pm}) := -\mathrm{i} \sum_{j=1}^{n} \frac{\partial}{\partial p_j} \big(e_{j\,1}^j r_{23}^{\pm}(p) - e_{j\,2}^j r_{13}^{\pm}(p) + e_{j\,3}^j r_{12}^{\pm}(p)\big) \equiv \mathrm{Alt}(\mathrm{d}r). \tag{3.39}$$

As the verification of (3.38) requires some work, we sketch the main steps in the appendix.

4. Quantization

4.1. Quantum exchange relations and their quasiclassical limit

The exchange relations for the quantum matrix algebra—which we shall again denote by $\mathcal{M}_q$—were derived earlier on the basis of an analysis of the braiding properties of $SU(n)_k$ WZNW 4-point blocks [32, 39] satisfying the Knizhnik–Zamolodchikov equations [41, 48]. They have the form [32]

$$[q^{p_{ij}}, q^{p_{k\ell}}] = 0 \qquad q^{p_{ij}} a_\alpha^\ell = a_\alpha^\ell q^{p_{ij} + \delta_i^\ell - \delta_j^\ell} \tag{4.1}$$

$$\hat{R}(p)^{\pm 1} a_1 a_2 = a_1 a_2 \hat{R}^{\pm 1} \tag{4.2}$$

where

$$\left(q^{\frac{1}{n}}\hat{R}\right)^{\pm 1}_{i\,i+1} = q^{\pm 1}\mathbb{1}_{i\,i+1} - A_{i\,i+1} \qquad A^{\alpha_1\alpha_2}_{\beta_1\beta_2} = q^{\epsilon_{\alpha_2\alpha_1}}\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2} - \delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1} \tag{4.3}$$

$$q^{\epsilon_{\alpha_1\alpha_2}} = \begin{cases} q^{-1} & \text{for} \quad \alpha_1 < \alpha_2 \\ 1 & \text{for} \quad \alpha_1 = \alpha_2 \\ q & \text{for} \quad \alpha_1 > \alpha_2 \end{cases} \qquad q = \mathrm{e}^{-\mathrm{i}\frac{\pi}{h}} \qquad h = k + n \tag{4.4}$$

$$\left(q^{\frac{1}{n}}\hat{R}(p)\right)^{\pm 1}_{i\,i+1} = q^{\pm 1}\mathbb{1}_{i\,i+1} - A_{i\,i+1}(p) \qquad A^{i_1 i_2}_{j_1 j_2}(p) = \frac{[p_{i_1 i_2} - 1]}{[p_{i_1 i_2}]}\left(\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} - \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\right). \tag{4.5}$$

Both $A_{i\,i+1} =: A_i$ and $A_{i\,i+1}(p) =: A_i(p)$ satisfy the Hecke algebra relations

$$A_i A_{i+1} A_i - A_i = A_{i+1} A_i A_{i+1} - A_{i+1} \qquad A_i^2 = [2] A_i$$
$$[A_i, A_j] = 0 \qquad \text{for} \quad |i - j| > 1. \tag{4.6}$$

It remains to verify that the quasiclassical limit of these relations indeed reproduces the PB relations of section 3.

One can introduce two deformation parameters: $\frac{1}{k}$ and the (implicit in common notation) Planck constant $\hbar$ (see [1]). If one ascribes to the physical quantities $\tilde{k}$ and $\tilde{p}$ the dimension of action, then our dimensionless numbers k and p shall be written as $k = \frac{\tilde{k}}{\hbar}$ and $p = \frac{\tilde{p}}{\hbar}$. We shall distinguish the quasiclassical limit ($\hbar \to 0$) from the undeformed limit ($k \to \infty$) without using the parameter $\hbar$, by characterizing the second one by

$$k \to \infty \qquad p_{j\ell} \text{ finite} \qquad \frac{p_{j\ell}}{k} \to 0 \tag{4.7}$$

while setting for the first one of interest

$$\frac{k}{n} \to \infty \qquad p_{j\ell} \to \infty \qquad \frac{p_{j\ell}}{k} \text{ finite} \qquad (j < \ell). \tag{4.8}$$

The substitution of the *level* k by the *height* $h = k + n$ in the quantum expression for q (4.4) is consistent with (4.8) but we are only aware of an explanation of its necessity that uses the full (with infinite number of degrees of freedom) WZNW model which involves the Sugawara formula expressing the stress energy tensor as a normal square of the $SU(n)$ current (see [41, 47]).

Let P be the permutation operator for either set of indices, $j, \ell, \ldots$ or $\alpha, \beta, \ldots$:

$$P_{12} = \left(P^{j_1 j_2}_{\ell_1 \ell_2}\right) = \left(\delta^{j_1}_{\ell_2}\delta^{j_2}_{\ell_1}\right) \qquad \text{or} \qquad P_{12} = \left(P^{\alpha_1\alpha_2}_{\beta_1\beta_2}\right) = \left(\delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1}\right) \tag{4.9}$$

and let $\mathbb{1}_{12}$ be the corresponding unit operator (e.g., $\mathbb{1}^{j_1 j_2}_{\ell_1 \ell_2} = \delta^{j_1}_{\ell_1}\delta^{j_2}_{\ell_2}$). Then we can write

$$R^{\alpha_1\alpha_2}_{\beta_1\beta_2} = (\hat{R}P)^{\alpha_1\alpha_2}_{\beta_1\beta_2} = \overline{q}^{\frac{1}{n}}\left((q - q^{\epsilon_{\alpha_2\alpha_1}})P^{\alpha_1\alpha_2}_{\beta_1\beta_2} + \mathbb{1}^{\alpha_1\alpha_2}_{\beta_1\beta_2}\right) \tag{4.10}$$

$$R(p)^{j_1 j_2}_{\ell_1 \ell_2} = (\hat{R}P)^{j_1 j_2}_{\ell_1 \ell_2} = \overline{q}^{\frac{1}{n}}\left(\frac{q^{p_{j_1 j_2}}}{[p_{j_1 j_2}]}P^{j_1 j_2}_{\ell_1 \ell_2} + \frac{[p_{j_1 j_2} - 1]}{[p_{j_1 j_2}]}\mathbb{1}^{j_1 j_2}_{\ell_1 \ell_2}\right). \tag{4.11}$$

Setting now

$$q = 1 - \mathrm{i}\frac{\pi}{k} + \mathcal{O}\left(\frac{\pi^2}{k^2}\right) \qquad \left(\overline{q}^{\frac{1}{n}} = 1 + \mathrm{i}\frac{\pi}{nk} + \mathcal{O}\left(\frac{\pi^2}{k^2}\right)\right)$$
$$\frac{[p-1]}{[p]} = 1 - \frac{\pi}{k}\cot\left(\frac{\pi}{k}p\right) + \mathcal{O}\left(\frac{\pi^2}{k^2}\right) \tag{4.12}$$

we find

$$R_{12} = \mathbb{1}_{12} + \mathrm{i}\frac{\pi}{k}r^-_{12} + \mathcal{O}\left(\frac{\pi^2}{k^2}\right) \qquad R(p)_{12} = \mathbb{1}_{12} + \mathrm{i}r^-_{12}(p) + \mathcal{O}\left(\frac{\pi^2}{k^2}\right) \tag{4.13}$$

where

$$r_{12}^{-} = r_{12} - C_{12} \qquad r_{\beta_1\beta_2}^{\alpha_1\alpha_2} = -\epsilon_{\alpha_1\alpha_2} P_{\beta_1\beta_2}^{\alpha_1\alpha_2} \qquad C_{12} = P_{12} - \frac{1}{n}\mathbb{1}_{12} \tag{4.14}$$

$$r_{12}^{-}(p) = r_{12} - C_{12} \qquad r(p)_{\ell_1\ell_2}^{j_1j_2} = \mathrm{i}\frac{\pi}{k}\cot\left(\frac{\pi}{k}p_{j_1j_2}\right)\left(\delta_{\ell_1}^{j_1}\delta_{\ell_2}^{j_2} - \delta_{\ell_2}^{j_1}\delta_{\ell_1}^{j_2}\right). \tag{4.15}$$

The reason why we are keeping the factor $\frac{\pi}{k}$ in the definition of $r_{12}(p)$ is that it has a nonzero undeformed limit since

$$\lim_{k\to\infty}\frac{\pi}{k}\operatorname{cotg}\left(\frac{\pi}{k}p\right) = \frac{1}{p}. \tag{4.16}$$

Taking into account that $[C_{12}, a_1a_2] = 0$, we thus recover the PB relations of section 3.2.

4.2. $U_q(s\ell_n)$ symmetry of the exchange relations

Let, for $G_0 \ni M \equiv \left(M_j^i\right)_{i,j=1}^n$, $M_n^n \neq 0 \neq \det\begin{pmatrix} M_{n-1}^{n-1} & M_n^{n-1} \\ M_{n-1}^n & M_n^n \end{pmatrix}$ etc and

$$M = q^{\frac{1}{n}-1}M_+M_-^{-1} \qquad M_+ = N_+D \qquad M_-^{-1} = N_-D \qquad D = \left(d_\alpha\delta_\beta^\alpha\right) \tag{4.17}$$

$$N_+ = \begin{pmatrix} 1 & f_1 & f_{12} & \dots \\ 0 & 1 & f_2 & \dots \\ 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \qquad N_- = \begin{pmatrix} 1 & 0 & 0 & \dots \\ e_1 & 1 & 0 & \dots \\ e_{21} & e_2 & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{4.18}$$

where the common diagonal matrix D has unit determinant: $d_1d_2\dots d_n = 1$. It can be deduced from (4.1), (4.2) and $M = a^{-1}M_pa$ that

$$[\hat{R}^{\pm}, M_{2\pm}M_{1\pm}] = 0 \qquad \hat{R}M_{2-}M_{1+} = M_{2+}M_{1-}\hat{R}. \tag{4.19}$$

It is known that equations (4.19) for the matrices $M_\pm$ are equivalent to the defining relations of the quantum universal enveloping algebra $U_q := U_q(s\ell_n)$ [16] that is paired by duality to $Fun(SL_q(n))$ [28]. The Chevalley generators of U_q are related to the elements of the matrices (4.17), (4.18) by ([28], see also [34])

$$\begin{aligned} &d_i = q^{\Lambda_{i-1}-\Lambda_i} \qquad (i = 1,\dots,n,\ \Lambda_0 = 0 = \Lambda_n) \\ &e_i = (\overline{q}-q)E_i \qquad f_i = (\overline{q}-q)F_i \\ &(\overline{q}-q)f_{12} = f_2f_1 - qf_1f_2 = (\overline{q}-q)^2(F_2F_1 - qF_1F_2) \text{ etc} \\ &(\overline{q}-q)e_{21} = e_1e_2 - qe_2e_1 = (\overline{q}-q)^2(E_1E_2 - qE_2E_1) \text{ etc.} \end{aligned} \tag{4.20}$$

Here Λ_i are the fundamental co-weights of $s\ell(n)$ related to the co-roots H_i by $H_i = 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1}$; E_i and F_i are the raising and lowering operators satisfying

$$\begin{aligned} &[E_i, F_j] = [H_i]\delta_{ij} \qquad q^{\Lambda_i}E_j = E_jq^{\Lambda_i+\delta_{ij}} \qquad q^{\Lambda_i}F_j = F_jq^{\Lambda_i-\delta_{ij}} \\ &[E_i, E_j] = 0 = [F_i, F_j] \qquad \text{for} \quad |j-i| \geqslant 2 \\ &[2]X_iX_{i\pm1}X_i = X_{i\pm1}X_i^2 + X_i^2X_{i\pm1} \qquad \text{for} \quad X = E, F. \end{aligned} \tag{4.21}$$

The exchange relations (4.1), (4.2) imply

$$M_{1\pm} P a_1 = a_2 \hat{R}^{\mp 1} M_{2\pm} \tag{4.22}$$

(see [32]). It follows that these exchange relations are invariant under the coaction of U_q,

$$\begin{aligned} &\left[E_a, a_\alpha^i\right] = \delta_{a\alpha-1} a_{\alpha-1}^i q^{H_a} \qquad && \left[q^{H_a} F_a, a_\alpha^i\right] = \delta_{a\alpha} q^{H_a} a_{\alpha+1}^i \\ &q^{H_a} a_\alpha^i = a_\alpha^i q^{H_a + \delta_{a\alpha} - \delta_{a\alpha-1}} && a = 1, \ldots, n-1. \end{aligned} \tag{4.23}$$

We note that the centralizer of q^{p_i} $\left(\prod_{i=1}^n q^{p_i} = 1\right)$ in the algebra (4.1), (4.2) (i.e. the maximal subalgebra commuting with all q^{p_i}) is spanned by U_q over the field $\mathbb{Q}(q, q^{p_i})$ of rational functions of q^{p_i}.

4.3. Operator realization

We shall sketch and briefly discuss the finite-dimensional Fock-like space realization of the quantum matrix algebra of [32].

The 'Fock space' $\mathcal{F}$ and its dual $\mathcal{F}'$ are defined as $\mathcal{M}_q$-modules with one-dimensional U_q-invariant subspaces of multiples of (nonzero) bra and ket vacuum vectors $\langle 0|$ and $|0\rangle$ (such that $\langle 0|\mathcal{M}_q = \mathcal{F}'$, $\mathcal{M}_q|0\rangle = \mathcal{F}$) satisfying

$$\begin{aligned} &a_\alpha^i |0\rangle = 0 \qquad \text{for} \quad i > 1 \qquad && \langle 0| a_\alpha^j = 0 \qquad \text{for} \quad j < n \\ &q^{p_{ij}} |0\rangle = q^{j-i} |0\rangle && \langle 0| q^{p_{ij}} = q^{j-i} \langle 0| \\ &(X - \varepsilon(X))|0\rangle = 0 = \langle 0|(X - \varepsilon(X)) && \forall X \in U_q \end{aligned} \tag{4.24}$$

with $\varepsilon(X)$ the co-unit. The duality between $\mathcal{F}$ and $\mathcal{F}'$ is established by a bilinear pairing $\langle \cdot | \cdot \rangle$ such that

$$\langle 0|0\rangle = 1 \qquad \langle \Phi | A | \Psi \rangle = \langle \Psi | A' | \Phi \rangle \tag{4.25}$$

where $A \to A'$ is a linear anti-involution (*transposition*) of $\mathcal{M}_q$ defined for generic q by

$$\mathcal{D}_i(p) \left(a_\alpha^i\right)' = \tilde{a}_i^\alpha := \frac{1}{[n-1]!} \mathcal{E}^{\alpha\alpha_1\ldots\alpha_{n-1}} \varepsilon_{ii_1\ldots i_{n-1}} a_{\alpha_1}^{i_1} \ldots a_{\alpha_{n-1}}^{i_{n-1}} \qquad (q^{p_i})' = q^{p_i}. \tag{4.26}$$

Here $\mathcal{D}_i(p)$ stands for the product

$$\mathcal{D}_i(p) = \prod_{j<\ell, j\neq i\neq \ell} [p_{j\ell}] \qquad \left(\Rightarrow \left[\mathcal{D}_i(p), a_\alpha^i\right] = 0 = \left[\mathcal{D}_i(p), \tilde{a}_i^\alpha\right]\right). \tag{4.27}$$

For the definition of the U_q- and, respectively, the 'dynamical' Levi-Civita tensors $\mathcal{E}^{\alpha_1\alpha_2\ldots\alpha_n}$, $\varepsilon_{i_1 i_2 \ldots i_n}$ see [38, 32]. The anti-involution (4.26) extends the known transposition of U_q determined by its action on the Chevalley generators (see section 3 of [34]),

$$E_i{}' = F_i q^{H_i - 1} \qquad F_i{}' = q^{1-H_i} E_i \qquad (q^{H_i})' = q^{H_i} \tag{4.28}$$

to the quantum matrix algebra (cf section 3.1 and appendix B of [32]).

The space $\mathcal{F}$ admits a canonical basis of weight vectors whose inner product can be computed (see section 3.2 of [32]). For $n = 2$ the basis has the simple form

$$|p, m\rangle = \left(a_1^1\right)^m \left(a_2^1\right)^{p-1-m} |0\rangle \qquad 0 \leqslant m \leqslant p-1 \qquad (p \equiv p_{12}) \tag{4.29}$$

and the inner product is given by

$$\langle p', m' | p, m \rangle = \delta_{pp'} \delta_{mm'} \overline{q}^{m(p-1-m)} [m]! [p-1-m]!. \tag{4.30}$$

For the deformation parameter q appearing in (4.4),

$$q = \mathrm{e}^{-\mathrm{i}\frac{\pi}{h}} \qquad h = k + n \quad \Rightarrow \quad q^h = -1 \tag{4.31}$$

i.e. q a (here, even) root of unity, the Fock space has an infinite-dimensional U_q invariant subspace of *null vectors* orthogonal to any vector in $\mathcal{F}$. In the $n = 2$ case all null vectors belong to the set $\mathcal{I}_h|0\rangle$ where $\mathcal{I}_h$ is the ideal generated by $[hp]$, $[hH]$, $q^{hp} + q^{hH}$, $\left(a^i_\alpha\right)^h$, $i, \alpha = 1, 2$. The definition of the ideal $\mathcal{I}_h$ can be generalized to any $n \geqslant 2$ assuming that it includes the hth powers of all minors of the quantum matrix $\left(a^i_\alpha\right)$. For $n = 2$ the factor space $\mathcal{F}_h$ is spanned by vectors of the form (4.29) with $0 < p < 2h$ and m in the range $0 \leqslant m \leqslant p - 1$, for $1 \leqslant p \leqslant h$, and $p - h \leqslant m \leqslant h - 1$ for $h + 1 \leqslant p \leqslant 2h - 1$. It splits into a direct sum of $2h - 1$ irreducible representations of $U_q(s\ell_2)$ of total dimension h^2.

For general n and generic q (i.e. for q not a root of unity) the space $\mathcal{F}$ has been proved to be a model space for U_q (see section 3.1 of [32]). The question of what should be viewed as a model space for the *reduced* U_q (U_q factored by its maximal ideal) for q satisfying (4.31) appears to be unsettled. If we define it as the direct sum of *integrable representations* (those with $0 < p < h$, for $n = 2$) of multiplicity 1, then the question arises whether there is a natural (say, a BRS type) procedure that would reduce $\mathcal{F}_h$ to such a sum. A BRS procedure was introduced in [22, 23] for the tensor product of two copies of $\mathcal{F}_h$—corresponding to the left and right movers' zero modes of an $SU(2)$ WZNW model. It would be interesting to pursue a similar approach to the problem at hand.

5. Concluding remarks

We have tried to make the present study of the chiral zero modes' phase space reasonably self-contained and have, hence, included some known material. It may be, therefore, useful to list at this point what appears to us as the main new features in our treatment.

We find explicitly the correspondence between the WZ term $\rho(a^{-1}M_p a)$ (rendering the zero modes' symplectic form (2.10) closed) and the solutions of the CYBE.

It is essential for the present treatment of the $SU(n)$ case that the determinant $\det\left(a^i_\alpha\right)$ of the zero modes' $n \times n$ matrix is set equal to an (essentially unique) pseudoinvariant q-polynomial in the $su(n)$ weights (see (1.7)). Accordingly, the symplectic form (2.10) in the $(n-1)(n+2)$-dimensional zero modes' phase manifold $\mathcal{M}_q$ necessarily contains, for $n > 2$, a term $\omega_q(M_p)$ depending only on the diagonal monodromy.

The counterpart of $\omega_q(M_p)$ in the symplectic form of the chiral WZNW model with diagonal monodromy, being closed by itself, is often omitted. This additional term is necessary in order to reproduce upon quantization the basic exchange relations involving the dynamical R-matrix of [37, 1, 15, 32].

The expression for ω_q is simpler—and easier to derive—in the extended ($n(n+1)$-dimensional) phase space $\mathcal{M}^{ex}_q$ spanned by p_i and a^j_α, $i, j, \alpha = 1, \ldots, n$, the form ω^{ex}_q (2.28) being nontrivial even for $n = 2$ (yielding, in the undeformed limit, the standard symplectic structure on $\mathbb{C}^2$ viewed as a Kähler manifold in that case).

The Dirac brackets of the physically interesting quantities a^j_α and $p_{j\ell}$ coincide with their Poisson brackets since they Poisson commute with one of the constraints and can be, hence, derived working in the (more symmetric) extended phase space.

Expression (3.30) for the Poisson bivector in $\mathcal{M}_q^{ex}$ allows us to directly compute the Poisson brackets of interest.

The quantum theory of chiral zero modes has been only briefly reviewed in section 4 concluding with the formulation of an open problem related to the concept of a model space for the quantum universal enveloping algebra $U_q(s\ell_n)$ for q a root of unity.

Acknowledgments

PF acknowledges the support of the Italian Ministry of Education, University and Research (MIUR). LH thanks Dipartimento di Fisica Teorica (DFT) dell' Università di Trieste and Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Trieste; both he and IT also acknowledge the hospitality and support of the Erwin Schrödinger International Institute for Mathematical Physics (ESI), Vienna. This work is supported in part by the Bulgarian National Council for Scientific Research under contract F-828.

Appendix

We begin by reproducing the properties of the polarized Casimir operators C_{mn} relevant for the proof of the CYBE (for both constant and 'dynamical', i.e. p-dependent, $r^{\pm}$):

$$[C_{12}, C_{13} + C_{23}] = 0 = [C_{12} + C_{13}, C_{23}]. \tag{A.1}$$

For $G = SU(n)$, C_{12} is given, essentially, by the permutation operator (4.9):

$$C_{12} = P_{12} - \frac{1}{n}\mathbb{1}_{12} \qquad \text{or} \qquad C^{j_1 j_2}_{\ell_1 \ell_2} = \delta^{j_1 j_2}_{\ell_2 \ell_1} - \frac{1}{n}\delta^{j_1 j_2}_{\ell_1 \ell_2} \qquad \left(\delta^{jk}_{\ell m} := \delta^j_\ell \delta^k_m\right) \tag{A.2}$$

and equation (A.1) gives

$$[C_{12}, C_{13} + C_{23}] + [C_{13}, C_{23}] = -[C_{12}, C_{23}] = \left(\delta^{j_1 j_2 j_3}_{\ell_2 \ell_3 \ell_1} - \delta^{j_1 j_2 j_3}_{\ell_3 \ell_1 \ell_2}\right). \tag{A.3}$$

Next we verify that the mixed (r–C) terms in the CYBE (3.38) (or (3.10)) vanish,

$$[r_{12}(p), C_{13} + C_{23}] + [r_{12}(p), C_{23} - C_{12}] + [C_{12} + C_{13}, r_{23}(p)] = 0 \tag{A.4}$$

using, e.g., the general identities $P_{ab} r_{bc} = r_{ac} P_{ab}$ for a, b, c all different, as well as skewsymmetry of $r_{ab} = -r_{ba}$. Computing the sum of commutators in (3.38), we find

$$\begin{aligned} &([r_{12}(p), r_{13}(p) + r_{23}(p)] + [r_{13}(p), r_{23}(p)])^{j_1 j_2 j_3}_{\ell_1 \ell_2 \ell_3} \\ &\quad = \frac{\pi^2}{k^2}\left(\left(c_{j_1 j_2} + c_{j_2 j_3}\right) c_{j_1 j_3} - c_{j_1 j_2} c_{j_2 j_3}\right)\left(\delta^{j_1 j_2 j_3}_{\ell_2 \ell_3 \ell_1} - \delta^{j_1 j_2 j_3}_{\ell_3 \ell_1 \ell_2}\right) \end{aligned} \tag{A.5}$$

where

$$c_{j\ell} := \cot\frac{\pi}{k} p_{j\ell} = -c_{\ell j} \qquad j \neq \ell \qquad c_{\ell\ell} := 0. \tag{A.6}$$

On the other hand, (3.39) gives

$$\mathrm{Alt}(\mathrm{d}r) = \frac{\pi}{k}\left(\delta_{j_1 j_2} c'_{j_2 j_3} + \delta_{j_2 j_3} c'_{j_1 j_3} + \delta_{j_1 j_3} c'_{j_1 j_2}\right)\left(\delta^{j_1 j_2 j_3}_{\ell_2 \ell_3 \ell_1} - \delta^{j_1 j_2 j_3}_{\ell_3 \ell_1 \ell_2}\right) \tag{A.7}$$

where

$$c'_{j\ell} := -\frac{\pi}{k}\frac{1}{\sin^2 \frac{\pi}{k} p_{j\ell}} = c'_{\ell j} \qquad j \neq \ell \qquad c'_{\ell\ell} := 0. \tag{A.8}$$

To prove (3.38), it suffices to combine (A.3)–(A.8) with one of the following relations (depending on whether all three indices j_1, j_2, j_3 are different or not):

$$(\cot\alpha + \cot\beta)\cot(\alpha+\beta) - \cot\alpha\cot\beta = -1 \tag{A.9}$$

$$\cot^2\alpha - \frac{1}{\sin^2\alpha} = -1. \tag{A.10}$$

References

[1] Alekseev A Yu and Faddeev L D 1991 $(T^*G)_t$: a toy model for conformal field theory *Commun. Math. Phys.* **141** 413–22

[2] Alekseev A Yu, Kosmann-Schwarzbach Y and Meinrenken E 2002 Quasi-Poisson manifolds *Can. J. Math.* **54** 3–29 (*Preprint* math.DG/0006168)

[3] Alekseev A Yu and Meinrenken E 2002 Clifford algebras and the classical dynamical Yang–Baxter equation *Preprint* math.RT/0209347

[4] Alekseev A Yu and Todorov I T 1994 Quadratic brackets from symplectic forms *Nucl. Phys.* B **421** 413–28 (*Preprint* hep-th/9307026)

[5] Babelon O 1988 Extended conformal algebras and the Yang–Baxter equation *Phys. Lett.* B **215** 523–9

[6] Babelon O 1991 Universal exchange algebra for Bloch waves and Liouville theory *Commun. Math. Phys.* **139** 619–49

[7] Babelon O, Bernard D and Billey E 1996 A quasi-Hopf algebra interpretation of quantum 3-j and 6-j symbols and difference equations *Phys. Lett.* B **375** 89–97 (*Preprint* q-alg/9511019)

[8] Balog J, Dąbrowski L and Fehér L 1990 Classical r-matrix and exchange algebra in WZNW and Toda theories *Phys. Lett.* B **244** 227–34

[9] Balog J, Fehér L and Palla L 1999 The chiral WZNW phase space and its Poisson–Lie groupoid *Phys. Lett.* B **463** 83–92 (*Preprint* hep-th/ 9907050)

[10] Balog J, Fehér L and Palla L 2000 Chiral extensions of the WZNW phase space, Poisson–Lie symmetries and groupoids *Nucl. Phys.* B **568** 503–42 (*Preprint* hep-th/9910046)

[11] Balog J, Fehér L and Palla L 2000 The chiral WZNW phase space as a quasi-Poisson space *Phys. Lett.* A **277** 107–14 (*Preprint* hep-th/0007045)

[12] Bargmann V 1962 On the representations of the rotation group *Rev. Mod. Phys.* **34** 820–45

[13] Belavin A A and Drinfeld V G 1982 Solutions of the classical Yang–Baxter equations for simple Lie algebras *Funct. Anal. Appl.* **16** 159–80

[14] Blok B 1989 Classical exchange algebra in the Wess–Zumino–Witten model *Phys. Lett.* B **233** 359–62

[15] Bytsko A G and Faddeev L D 1996 $(T^*B)_q$, q-analogue of model space and the CGC generating matrices *J. Math. Phys.* **37** 6324–48 (*Preprint* q-alg/9508022)

[16] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)

[17] Chu M, Goddard P, Halliday I, Olive D and Schwimmer A 1991 Quantization of the Wess–Zumino–Witten model on a circle *Phys. Lett.* B **266** 71–81

[18] Chu M and Goddard P 1995 Quantization of the $SU(n)$ WZW model at level k *Nucl. Phys.* B **445** 145–68 (*Preprint* hep-th/9407116)

[19] Dąbrowski L and Reina C 2002 Quantum spin coverings and statistics *Preprint* math.QA/0208088

[20] Dirac P A M 1950 Generalized Hamiltonian dynamics *Canad. J. Math.* **2** 129–48

[21] Dubois-Violette M, Furlan P, Hadjiivanov L K, Isaev A P, Pyatov P N and Todorov I T 1999 A finite dimensional gauge problem in the WZNW model *Quantum Theory and Symmetries. Proc. Int. Symp. (Goslar, Germany, 18–22 July 1999)* ed H-D Doebner and V Dobrev (*Preprint* hep-th/9910206)

[22] Dubois-Violette M and Todorov I T 1997 Ceneralized cohomologies for the zero modes of the $SU(2)$ WZNW model *Lett. Math. Phys.* **42** 183–92 (*Preprint* hep-th/9704069)

[23] Dubois-Violette M and Todorov I T 1999 Generalized homologies for the zero modes of the $SU(2)$ WZNW model *Lett. Math. Phys.* **48** 323–38 (*Preprint* math.QA/9905071)

[24] Etingof P 2002 On the dynamical Yang–Baxter equation *Preprint* math.QA/0207008

[25] Etingof P and Schiffmann O 2000 On the moduli space of classical dynamical r-matrices *Math. Phys. Lett.* **8** 157–70

[26] Etingof P and Varchenko A 1998 Geometry and classification of solutions of the classical dynamical Yang–Baxter equation *Commun. Math. Phys.* **192** 77–120 (*Preprint* q-alg/9703040)

[27] Faddeev L D 1990 On the exchange matrix for WZNW model *Commun. Math. Phys.* **132** 131–8

[28] Faddeev L D, Reshetikhin N Yu and Takhtajan L A 1989 Quantization of Lie groups and Lie algebras *Algebra Anal.* **1** 178–206 (Engl. transl. 1990 *Leningrad Math. J.* **1**, 193–225)

[29] Falceto F and Gawędzki K 1993 Lattice Wess–Zumino–Witten model and quantum groups *J. Geom. Phys.* **11** 251–79

Falceto F and Gawędzki K 1991 *Quantum Group Symmetries in WZW Models. Unpublished Notes* (Bures-sur-Yvette: IHES)

[30] Fehér L and Gábor A 2001 On interpretations and constructions of classical dynamical r-matrices *Preprint* hep-th/0111252

[31] Felder G 1995 Conformal field theory and integrable systems associated to elliptic curves *Proc. ICM (Zürich 1994)* ed S D Chatterji (Basel: Birkhauser) pp 1247–55 (*Preprint* hep-th/9407154)

[32] Furlan P, Hadjiivanov L, Isaev A P, Ogievetsky O V, Pyatov P N and Todorov I 2000 Quantum matrix algebra for the $SU(n)$ WZNW model *Preprint* hep-th/0003210

[33] Furlan P, Hadjiivanov L K and Todorov I T 1996 Operator realization of the $SU(2)$ WZNW model *Nucl. Phys.* B **474** 497–511 (*Preprint* hep-th/9602101)

[34] Furlan P, Hadjiivanov L and Todorov I 1997 A quantum gauge group approach to the $2D$ $SU(n)$ WZNW model *Int. J. Mod. Phys.* A **12** 23–32 (*Preprint* hep-th/9610202)

[35] Gawędzki K 1991 Classical origin of quantum group symmetries in Wess–Zumino–Witten conformal field theory *Commun. Math. Phys.* **139** 201–13

[36] Gawędzki K, Todorov I T and Tran-Ngoc-Bich P 2001 Canonical quantization of the boundary Wess–Zumino–Witten model *Preprint* hep-th/0101170

[37] Gervais J-L and Neveu A 1984 Novel triangle relation and absence of tachions in Liouville theory *Nucl. Phys.* B **238** 125–41

[38] Hadjiivanov L K, Isaev A P, Ogievetsky O V, Pyatov P N and Todorov I T 1999 Hecke algebraic properties of dynamical R-matrices. Application to related matrix algebras *J. Math. Phys.* **40** 427–48 (*Preprint* q-alg/9712026)

[39] Hadjiivanov L K, Stanev Ya S and Todorov I T 2000 Regular basis and R-matrices for the $\widehat{su}(n)_k$ Knizhnik–Zamolodchikov equation *Lett. Math. Phys.* **54** 137–55 *Preprint* hep-th/0007187

[40] Julia B and Silva S 2002 On covariant phase space methods *Preprint* hep-th/0205072

[41] Knizhnik V G and Zamolodchikov A B 1984 Current algebra and Wess–Zumino model in two dimensions *Nucl. Phys.* B **247** 83–103

[42] Lu J-H 1999 Classical dynamical r-matrices and homogeneous Poisson structures on G/H and K/T *Preprint* math.SG/9909004

[43] Liu Z-J and Xu P 1999 Dirac structures and dynamical r-matrices *Preprint* math.DG/9903119

[44] Novikov S P 1982 The Hamiltonian formalism and a multivalued analogue of Morse theory *Usp. Mat. Nauk* **37** 3–49 (Engl. transl. 1982 *Russ. Math. Surveys* **37** 1–56)

[45] Semenov-Tian-Shansky M A 1985 Dressing transformations and Poisson group actions *Publ. RIMS, Kyoto Univ.* **21** 1237–60

Semenov-Tyan-Shanskii M A 1983 What is a classical R-matrix? *Funct. Anal. Appl.* **17** 259–72

[46] Schwinger J 1952 On angular momentum *US At. Energy Comm.* NYO-3071 (reprinted in Biedenharn L C and van Dam H (ed) 1965 *Quantum Theory of Angular Momentum* (New York: Academic) pp 229–79)

[47] Todorov I T 1986 Infinite Lie algebras in 2-dimensional conformal field theory *Proc. 12th Int. Conf. on Differential Geometric Methods in Physics (Shumen, Bulgaria, 1984)* ed H-D Doebner and T Palev (Singapore: World Scientific) pp 297–347

Todorov I T 1985 Current algebra approach to conformal invariant two-dimensional models *Phys. Lett.* B **153** 77–81

[48] Tsuchiya A and Kanie Y 1987 Vertex operators in the conformal field theory on $\mathbb{P}^1$ and monodromy representations of the braid group *Lett. Math. Phys.* **13** 303–12

[49] Wess J and Zumino B 1971 Consequences of anomalous Ward identities *Phys. Lett.* B **37** 95–7

[50] Witten E 1984 Non-abelian bosonization in two dimensions *Commun. Math. Phys.* **92** 455–72